Exact Limit of the Expected Periodogram in the Unit-Root Case

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21. September 2007

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September 21, 2007

Abstract

We derive the limit of the expected periodogram in the unit-root case under general conditions. This function is seen to be time-independent, thus sharing a fundamental property with the stationary case equivalent. We discuss the consequences of this result to the frequency domain interpretation of filtered integrated time series.

JEL Classification: C22

Keywords: Periodogram, Unit root

1 Motivation

Solo (1992) has shown that certain continuous-time stationary increment processes possess many of the frequency domain properties of stationary processes. Crucially, although their variance is

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infinite or time-varying (depending on the specification of initial conditions), they have a time-invariant spectrum, defined there as the limit of the expected periodogram. This more general definition of spectrum helps us understand the frequency domain properties of certain non-stationary processes, circumventing the restrictive nature of the standard spectral representation theorems for stationary processes. Crucially, and this is ou main concern, it sheds light on the frequency domain interpretation of the effects of applying trend extraction filters to integrated time series.

We show in this paper that Solo's (1992) main result holds in the case of (discrete-time) time series processes containing one unit root. Under very general conditions, we provide exact expressions for the time-invariant spectrum of an integrated time series, defined as the limit of the expected periodogram. It is shown that this limit differs from the commonly defined (pseudo-) spectrum of an integrated time series (e.g., as in Harvey 1993; Hurvich and Ray 1995; Young, Pedregal and Tych 1999; Velasco 1999; Phillips 1999; Den Haan and Sumner 2004). We will discuss the nuisance that this fact represents to the interpretation of the consequences of applying linear filters that render the series stationary.

2 Result

Denote $I_{T,x}(\omega_j)$ as the periodogram of the sequence $\{x_t\}_{t=1}^{T}$, where $\omega_j = 2\pi j/T$ are the integer multiples of $2\pi/T$ that fall in the interval $]-\pi,\pi]$. Restricting ourselves to real sequences and noting that $I_{T,x}(\omega_j) = I_{T,x}(-\omega_j)$ in this case, we extend as usual the periodogram for every frequency in the interval $[-\pi,\pi]$ in the following way:

$$I_{T,x}(\omega) = \begin{cases} I_{T,x}(\omega_k), \omega_k - \pi/T < \omega \leq \omega_k + \pi/T \\ I_{T,x}(-\omega), 0 < \omega \leq \pi \end{cases}$$

For $\omega \in [0, \pi]$, let $g(T, \omega)$ be the multiple of $2\pi/T$ closest to ω. If $\omega \in [-\pi, 0[$, let $g(T, \omega) = g(T, -\omega)$. Then,

$$I_{T,x}(\omega) = I_{T,x}(g(T, \omega)) \tag{1}$$

If $\{x_t\}_{t=1}^{T}$ is a sample from a stationary time series with mean μ and the autocovariance function $\gamma(.)$ is absolutely summable, it can be shown (see, e.g., Brockwell and Davis,1991, p.343) that:

$$E[I_{T,x}(0) - T\mu^2] \rightarrow 2\pi S_x(0) \text{ as } T \rightarrow \infty \tag{2}$$

$$E[I_{T,x}(\omega)] \rightarrow 2\pi S_x(\omega) \text{ as } T \rightarrow \infty, \omega \neq 0$$

where $S_x(\omega)$ is the spectrum of x_t. That is, when the sample size grows the periodogram converges to the distribution of variance as revealed by the spectral representation theorem. As Solo (1992), in the analysis of the spectrum of continuous-time, stationary increments processes, we argue that the result in (2) is a less restrictive inversion relation than that implied by the spectral representation theorem. The question that we address is whether or not the relation in (2) remains valid in the case of integrated processes. Does the expected value of the periodogram of an integrated series, which can be seen as a distribution of power, converge to a time-invariant function? The surprising answer is that it does, at least if the order of integration is 1 and for a very broad class of stationary increments. This is summarised in theorem 1.

Theorem 1. *Let* $u_t = \psi(L)\varepsilon_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$, *where* $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{\frac{1}{2}} < \infty$ *and* $\{\varepsilon_t\}$ *is a white noise sequence such that* $E[\varepsilon_t] = 0$ *and* $Var[\varepsilon_t] = \sigma_\varepsilon^2 < \infty$. *Consider the process* $\{x_t\}$ *verifying* $x_t - x_{t-1} = u_t, \forall t$. *Then, the periodogram of* x_t, $I_{T,x}(\omega)$, *has the following properties:*

$$i) \;\; T^{-2}E[I_{T,x}(0)] \rightarrow \frac{2\pi}{3} S_{\Delta x}(0) \text{ as } T \rightarrow \infty, \textit{ assuming } x_0 = 0$$

$$ii) \;\; E[I_{T,x}(\omega)] \rightarrow 2\pi S_x(\omega) \text{ as } T \rightarrow \infty, \omega \neq 0 \tag{3}$$

where $S_{\Delta x}(0)$ is the spectrum of $x_t - x_{t-1} = u_t$ at zero frequency and

$$S_x(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\psi(e^{-i\omega})|^2 + |\psi(1)|^2}{|1 - e^{-i\omega}|^2}, \omega \neq 0 \tag{4}$$

Proof: Consider first the case $\omega \neq 0$. Fix any $\omega \in]0, \pi]$. Then by (1) $I_{T,x}(\omega) = I_{T,x}(\omega_j)$ for some Fourier frequency ω_j. The discrete Fourier transform of $x_t - x_{t-1} = \Delta x_t$, denoted by $J_{T,\Delta x}(\omega_j)$, can be decomposed in the following way (as in Phillips 1999, but not fixing $x_0 = 0$):

$$J_{T,\Delta x}(\omega_j) = T^{-\frac{1}{2}} \sum_{t=1}^{T} \Delta x_t e^{-i\omega_j t} =$$

$$= T^{-\frac{1}{2}}(1 - e^{-i\omega_j}) \sum_{t=1}^{T} x_t e^{-i\omega_j t} + T^{-\frac{1}{2}} \sum_{t=1}^{T} x_t e^{-i\omega_j (t+1)} - T^{-\frac{1}{2}} \sum_{t=1}^{T} x_{t-1} e^{-i\omega_j t} =$$

$$= (1 - e^{-i\omega_j}) J_{T,x}(\omega_j) + T^{-\frac{1}{2}}(x_T e^{-i\omega_j (T+1)} - x_0 e^{-i\omega_j}). \tag{5}$$

where $J_{T,x}(\omega_j)$ denotes the discrete Fourier transform of x_t. Now, the periodogram of Δx_t can be written as $I_{T,\Delta x}(\omega_j) = J_{T,\Delta x}(\omega_j) J_{T,\Delta x}(-\omega_j)$. Multiplying both sides of (5) by $J_{T,\Delta x}(-\omega_j)$, using the fact that $e^{-i\omega_j (T+1)} = e^{-i\omega_j}$ for the Fourier frequencies ω_j and rearranging terms we get:

$$|1 - e^{-i\omega_j}|^2 I_{T,x}(\omega_j) = I_{T,\Delta x}(\omega_j) + T^{-1}(x_T - x_0)^2 -$$

$$-J_{T,\Delta x}(\omega_j) T^{-\frac{1}{2}}(x_T - x_0) e^{-i\omega_j} - J_{T,\Delta x}(-\omega_j) T^{-\frac{1}{2}}(x_T - x_0) e^{i\omega_j} \tag{6}$$

Now put $R_T(\omega_j) = J_{T,\Delta x}(\omega_j) T^{-\frac{1}{2}}(x_T - x_0) e^{-i\omega_j}$. Taking expectations we get:

$$E[R_T(\omega_j)] = T^{-1} e^{-i\omega_j} E[(\sum_{t=1}^{T} u_t e^{-i\omega_j t}) \sum_{t=1}^{T} u_t] = T^{-1} e^{-i\omega_j} \mathbf{1}' E[\mathbf{u}\mathbf{u}'] \mathbf{e}$$

where $\mathbf{1}$ is a vector of ones, $\mathbf{e} = (e^{-i\omega_j}, e^{-2i\omega_j}, ..., e^{-Ti\omega_j})'$ and $\mathbf{u} = (u_1, u_2, ..., u_T)'$. Since $E[\mathbf{u}\mathbf{u}'] =$

$[\gamma_{\Delta x}(j-i)]_{i,j=1}^{T}$, where $\gamma_{\Delta x}(.)$ is the autocovariance function of Δx_t, we get finally:

$$E[R_T(\omega_j)] = T^{-1}e^{-i\omega_j}\sum_{l=1}^{T}\sum_{h=1}^{T}\gamma_{\Delta x}(h-l)e^{-i\omega_j h}$$

This can be decomposed as follows:

$$E[R_T(\omega_j)] = T^{-1}e^{-i\omega_j}(\sum_{h=0}^{T-1}\gamma_{\Delta x}(h)e^{-i\omega_j h}\sum_{l=1}^{T-h}e^{-i\omega_j l} + \sum_{h=-T+1}^{-1}\gamma_{\Delta x}(h)e^{-i\omega_j h}\sum_{l=1-h}^{T}e^{-i\omega_j l}) \qquad (7)$$

Now, for $0 \leq h \leq T-1$ we have:

$$|\sum_{l=1}^{T-h}e^{-i\omega_j l}| = |\sum_{l=1}^{T}e^{-i\omega_j l} - \sum_{l=T-h+1}^{T}e^{-i\omega_j l}| = |0 - \sum_{l=T-h+1}^{T}e^{-i\omega_j l}| \leq h$$

since $\sum_{h=1}^{T}e^{-i\omega_j h} = \frac{1-e^{-i\omega_j T}}{1-e^{-i\omega_j}}e^{-i\omega_j} = 0$, as $e^{-i\omega_j T}=1$ for the Fourier frequencies ω_j. The inequality follows from the fact that $\omega_j \neq 0$ or 2π. Also, for $-T+1 \leq h \leq -1$ we can conclude that:

$$|\sum_{l=1-h}^{T}e^{-i\omega_j l}| \leq |h|$$

All this means that we can bound (7) by

$$T^{-1}\sum_{|h|<T}|\gamma_{\Delta x}(h)||h| \leq T^{-1}\sum_{|h|<T}\sum_{j=-\infty}^{\infty}|\psi_j\psi_{j+h}||h|$$

$$\leq T^{-\frac{1}{2}}\sum_{|h|<T}\sum_{j=-\infty}^{\infty}|\psi_j\psi_{j+h}||h|^{\frac{1}{2}} \leq T^{-\frac{1}{2}}(\sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|\psi_j\psi_{j+h}||h+j|^{\frac{1}{2}} + \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|\psi_j\psi_{j+h}||j|^{\frac{1}{2}})$$

$$= 2T^{-\frac{1}{2}}(\sum_{h=-\infty}^{\infty}|\psi_h||h|^{\frac{1}{2}})(\sum_{j=-\infty}^{\infty}|\psi_j|) \to 0 \text{ as } T\to\infty$$

since both series are convergent. Performing the same exercise for $R_T(-\omega_j)$ we conclude that the expected value of the last two terms in (6) converges to 0:

$$E[R_T(\omega_j) + R_T(-\omega_j)] \to 0 \text{ as } T \to \infty \tag{8}$$

As for the second term in (6) we get:

$$E[T^{-1}(x_T - x_0)^2] = T^{-1}Var[\sum_{t=1}^{T} \Delta x_t] =$$

$$= \sum_{|k|<T} (1 - \frac{|k|}{T})\gamma_{\Delta x}(k) \to \sum_{k=-\infty}^{\infty} \gamma_{\Delta x}(k) = 2\pi S_{\Delta x}(0) \text{ as } T \to \infty \tag{9}$$

by the dominated convergence theorem. Using (8), (9), the fact that $I_{T,\Delta x}(\omega) \to 2\pi S_{\Delta x}(\omega)$ as $T \to \infty$ (see (2)) and finally the fact that $|1 - e^{-ig(T,\omega)}|^2 \to |1 - e^{-i\omega}|^2$ as $T \to \infty$ (since $g(T, \omega) \to \omega$) we conclude that:

$$E[\frac{1}{2\pi} I_{T,x}(\omega)] \to \frac{S_{\Delta x}(\omega) + S_{\Delta x}(0)}{|1 - e^{-i\omega}|^2} = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\psi(e^{-i\omega})|^2 + |\psi(1)|^2}{|1 - e^{-i\omega}|^2}, \text{ as } T \to \infty, \ \omega \neq 0$$

which is time-invariant. For $\omega = 0$, we need to normalise the periodogram by T^3 instead of T, and also to take into account the initial condition $x_0 = 0$, which is equivalent to analyse the periodogram for $\{x_t - x_0\}$ instead of $\{x_t\}$. We get:

$$T^{-2}E[I_{T,x}(0)] = E[T^{-3}\sum_{t=1}^{T}(x_t - x_0)\sum_{t=1}^{T}(x_t - x_0)] = E[T^{-3}\sum_{t=1}^{T}\sum_{l=1}^{t} u_l \sum_{t=1}^{T}\sum_{l=1}^{t} u_l] = \mathbf{1}' E[\mathbf{u}_c \mathbf{u}_c'] \mathbf{1}$$

where $\mathbf{1}$ is a vector of ones and $\mathbf{u}_c = (u_1, u_1 + u_2, ..., \sum_{l=1}^{T} u_l)$. Evaluating $E[\mathbf{u}_c \mathbf{u}_c']$ we conclude that:

$$T^{-2}E[I_{T,x}(0)] = T^{-3} \sum_{|k|<T} \gamma_{\Delta x}(k) \sum_{h=1}^{T-|k|} (|k| + h)h$$

But:

$$\sum_{h=1}^{T-|k|}(|k|+h)h = \sum_{h=1}^{T-|k|}h^2 + |k|\sum_{h=1}^{T-|k|}h =$$

$$= \frac{(T-|k|)(T-|k|+1)(2(T-|k|)+1)}{6} + \frac{(T-|k|)(T-|k|+1)|k|}{2}$$

Thus,

$$T^{-2}E[I_{T,x}(0)] = \frac{1}{3}\sum_{|k|<T}\gamma_{\Delta x}(k)R(T,|k|)$$

where, for fixed $|k|$, $\lim\limits_{T\to\infty} R(T,|k|) = 1$. From the dominated convergence theorem, we finally conclude:

$$T^{-2}E[I_{T,x}(0)] \to \frac{2\pi}{3}S_{\Delta x}(0) \text{ as } T \to \infty$$

■

Example: Random walk. If $\{x_t\}$ verifies $x_t - x_{t-1} = \varepsilon_t, \forall t$ where $\{\varepsilon_t\}$ is a white noise sequence such that $E[\varepsilon_t] = 0$ and $Var[\varepsilon_t] = \sigma_\varepsilon^2$ we have, since $\psi(e^{-i\omega}) = \psi(1) = 1$:

$$S_x(\omega) = \frac{\sigma_\varepsilon^2}{\pi|1-e^{-i\omega}|^2}, \omega \neq 0$$

which shows that the pseudo-spectrum, defined as in theorem 1, is just proportional to the inverse of the Fourier transform of the differencing operator $(1-L)$ where L is the lag operator. However, if we apply the first difference filter to $\{x_t\}$ the spectrum of $(1-L)x_t = \varepsilon_t$ is given by $S_\varepsilon(\omega) = \sigma_\varepsilon^2/2\pi$. To perfectly maintain the relation $S_\varepsilon(\omega) = |1-e^{-i\omega}|^2 S_x(\omega)$ as in the stationary case we would need to define the pseudo-spectrum of x_t as:

$$S_x(\omega) = \frac{\sigma_\varepsilon^2}{2\pi|1-e^{-i\omega}|^2}, \omega \neq 0$$

which seems a neutral normalization of the (non-integrable) power distribution of x_t. In this case the first difference filter maintains the usual interpretation, summarised by the function $|1-e^{-i\omega}|^2$.

It attenuates low frequencies and amplifies high frequencies, thus producing a "noisier" output series. Now fix $x_0 = 0$ and $\sigma_\varepsilon^2 = 1$. The periodogram of x_t can be written as follows:

$$I_{x,T}(\omega_j) = T^{-1}|\sum_{t=1}^{T} x_t e^{itw_j}|^2 = \sum_{|k|<T} T^{-1} \sum_{t=1}^{T-|k|} x_t x_{t+|k|} e^{-ik\omega_j}$$

Next, fix any frequency $\omega \in]0, \pi]$. Theorem 1 shows that:

$$E[I_{T,x}(\omega)] = \sum_{|k|<T} T^{-1} \cos[g(T,\omega)k] \sum_{t=1}^{T-|k|} t =$$

$$= \frac{1}{2} \sum_{|k|<T} T^{-1} \cos[g(T,\omega)k](T - |k|)(T - |k| + 1) \to \frac{1}{\pi|1 - e^{-i\omega}|^2} \text{ as } T \to \infty$$

■

Remark 1. Except for $\omega = 0$, the convergence result of theorem 1 does not depend on any initial condition for x_0. The condition $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{\frac{1}{2}} < \infty$ is almost always used in a unit-root context to guarantee that the partial sums of u_t satisfy a functional central limit theorem but can be relaxed. It is easy to check that the proof works with $\sum_{j=-\infty}^{\infty} |\psi_j||j|^{\alpha} < \infty$ for some (small) $\alpha > 0$.

■

Remark 2. A different normalisation is needed for convergence if the order of integration is greater than 1. Consider the simplest case $(1 - L)^2 x_t = \varepsilon_t, \forall t$ where $\{\varepsilon_t\}$ is a white noise sequence. Performing the same calculations as in Example 1, again with $x_0 = 0$ and $\sigma_\varepsilon^2 = 1$, we obtain:

$$E[I_{T,x}(\omega)] = \sum_{|k|<T} T^{-1} \cos[g(T,\omega)k] \sum_{t=1}^{T-|k|} t(|k| + t)$$

which diverges since $\sum_{t=1}^{T-|k|} t(|k| + t)$ is a polynomial of order 3 in T. We shall not pursue any frequency domain characterisation in this case.

■

Theorem 1 is an adaptation of the continuous-time result in Solo (1992). Also, it sharpens the result of theorem 4 in Crato (1996) which gives an upper bound greater than 0 to the limit of (7). In a fractional integration context including unit roots, Hurvich and Ray (1995) have studied the behaviour of the expectation of the periodogram at Fourier frequencies close to the origin, obtaining also a time-invariance result. Specifically, theorem 1 in Hurvich and Ray (1995) shows the following, for a unit-root process:

$$E[\frac{1}{2\pi}I_{T,x}(\omega_j)/S_x^*(\omega_j)] \to 2 \text{ as } T \to \infty,\ \omega_j = 2\pi j/T \tag{10}$$

where

$$S_x^*(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{|\psi(e^{-i\omega})|^2}{|1-e^{-i\omega}|^2}, \omega \neq 0 \tag{11}$$

$S_x^*(\omega)$, which differs from $S_x(\omega)$ in theorem 1 (see discussion below), is interpreted as the spectrum of the integrated series as is in Velasco (1999) and Phillips (1999). Phillips (1999) argues that $S_x^*(\omega)$ has such interpretation in view of Solo's (1992) argument (i.e., $S_x^*(\omega)$ would be the limit of the expectation of the periodogram, which is not exactly true). It should be noted that j is held fixed, whereas our result is valid for any fixed $\omega \neq 0$. It is easy to reconcile the two results. Heuristically, once T grows, ω_j approaches 0 and hence $|\psi(e^{-i\omega_j})|^2$ approaches $|\psi(1)|^2$. Therefore $\frac{1}{2\pi}I_{T,x}(\omega_j)$ approaches $2S_x^*(\omega_j)$. In the stationary case the limit in (10) is just 1.

3 Interpreting filtered integrated time series

If we apply to the stationary sequence $\{x_t\}$ a time-invariant linear filter $h(L) = \sum_{j=-\infty}^{\infty} h_j L^j$, such that $\sum_{j=-\infty}^{\infty} |h_j| < \infty$ we obtain a filtered sequence $y_t = \sum_{k=-\infty}^{\infty} h_j x_{t-j}$. It is easy to verify that the spectrum of $\{y_t\}$ is given by:

$$S_y(\omega) = |h(e^{-i\omega})|^2 S_x(\omega) \tag{12}$$

where $S_x(\omega)$ is the spectrum of $\{x_t\}$ and $h(e^{-i\omega})$ is the transfer function of the filter. Can we extend the relation in (12) to integrated time series? This question is crucial when we want to interpret the effects of applying commonly used moving averages or simply the first difference filter to integrated time series. Common practice within this context is first to define the spectrum of an integrated process as the limit of the spectrum of a stationary process when the smallest autoregressive roots converge to 1 (e.g., Harvey 1993; Den Haan and Sumner 2004; Young, Pedregal and Tych, 1999). For a general ARIMA process the spectrum is defined as:

$$S_x^*(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\phi^{-1}(e^{-i\omega})|^2 |\theta(e^{-i\omega})|^2}{|1-e^{-i\omega}|^{2s}} = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\psi(e^{-i\omega})|^2}{|1-e^{-i\omega}|^{2s}}, \omega \neq 0 \tag{13}$$

where x_t satisfies:

$$\phi(L)(1-L)^s x_t = \theta(L)\varepsilon_t, \ \forall t$$

σ_ε^2 is the variance of the white-noise innovations ε_t, we assume the roots of $\phi(L)$ lie outside the unit circle and are different from those of $\theta(L)$, $\psi(L) = \phi(L)^{-1}\theta(L)$ and $s \geqslant 0$ the order of integration of the series. This limit is a time-invariant function at all frequencies except at those associated with autoregressive roots with unit modulus[1] and equals $S_x^*(\omega)$ in (11) when $s = 1$. An extension of the relation in (12) holds given the definition in (13), particularly when the filter renders the series stationary. It is assumed, without resorting to results such as that in theorem 1, that this function represents indeed a distribution of variance.

Bujosa, Bujosa and García-Ferrer (2002) provide a rigorous justification to the definition in (13), generalising the classical spectral analysis by developing an extended Fourier transform

[1]Since we assumed the roots of $\phi(L)$ lie outside the unit circle, we are only considering the existence of a pole at zero frequency. This assumption can straighforwardly be relaxed in order to include singularities at frequencies other than zero, e.g., due to non-stationary seasonal components.

to the field of fractions of polynomials. A pseudo-autocovariance generating function is defined and the corresponding extended Fourier transform is defined as the (pseudo-) spectrum of the integrated series, which leads to a functional form exactly as in (13). No representation theorem is provided but it is argued, again without stating a result such as that in theorem 1, that the usual interpretation of the spectrum as a decomposition of variance holds. Were the functions in theorem 1 (which only deals with one unit root) and in (13) the same for $\omega \neq 0$, one could state that defining the spectrum of an integrated series as the limit of the expected periodogram was a coherent extension of the stationary case inversion relation in (2). But the alert reader has noticed that the functional form in theorem 1 is slightly different than that in (13) due to the term $|\psi(1)|^2$ in the numerator. This is definitely a nuisance when the process is not a pure random walk, for which a straightforward normalisation (as in Example 1) preserves the power distribution and leads to the maintenance of the relation in (12). In any case, and given this normalisation, the differences in the interpretation would not be dramatic given the fact that the inverse of $|1 - e^{-i\omega}|^2$ dominates the behaviour of both functions at frequencies close to the pole located at zero frequency and the result in (10).

In short, one could be tempted to define the limit of the expected periodogram as the spectrum (or power distribution) of the integrated process, since in the stationary case this limit is the distribution of power revealed by the spectral representation theorem. However, this function differs slightly from the commonly defined (pseudo-) spectrum of an integrated time series, which has recently been given a rigorous interpretation by Bujosa, Bujosa and García-Ferrer (2002). Defining the spectrum of an integrated series as in theorem 1 would in general distort the interpretation given to the transfer function of filters applied to such series.

References

[1] Bujosa, A., Bujosa, M. and García-Ferrer (2002). A Note on Pseudo-spectra and Pseudo-Covariance Generating Functions of ARMA processes. Instituto Complutense de Análisis Económico Discussion Paper, Universidad Complutense de Madrid

[2] Crato, N. (1996). Some results on the spectral analysis of nonstationary time series. *Portugaliae Mathematica*, 53(2):179:186

[3] Den Haan, W.J., and Sumner, S.W. (2004). The Comovements between Real Activity and Prices in the G7, *European Economic Review*, 48:1333-1347

[4] Harvey, A.C. (1993). *Forecasting, structural time series models and the Kalman filter*. Cambridge University Press

[5] Hurvich, C.M. and Ray, B.K., (1995). Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes. *Journal of Time Series Analysis*, 16:17-42.

[6] Phillips, P.C.B. (1999). Discrete Fourier transforms of fractional processes. Cowles Foundation Discussion Paper, Yale University

[7] Solo, V.(1992). Intrinsic random functions and the paradox of 1/ f noise. *SIAM Journal of Applied Mathematics,* 52(1):270-291

[8] Velasco, C., (1999). Non-stationary log-periodogram regression. *Journal of Econometrics*, 91:325-371.

[9] Young, P. C., Pedregal, D. and Tych, W. (1999). Dynamic Harmonic Regression. *Journal of Forecasting*, 18:369-394.